



A class of models implying the Lambent–Jonas relation

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Abstract

A class of models describing penetration phenomenon is found which imply Lambent–Jonas correlation between the impact, the residual and the ballistic limit velocities. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Recht and Ipson (1963) derived the following formula for a high speed normal penetration assuming that a constant energy is absorbed by a target:

$$v_{\text{imp}}^n - av_{\text{res}}^n = v_{\text{bl}}^n, \quad v_{\text{imp}} \geq v_{\text{bl}} \quad (1)$$

Here v_{imp} is the impact (initial) velocity, v_{res} is the residual velocity, v_{bl} is the ballistic limit velocity (zero-residual initial velocity), parameter $n = 2$ and parameter a is given by the following formula:

$$a = \left(\frac{m_{\text{res}}}{m_{\text{imp}}} \right)^n \quad (2)$$

where m_{imp} and m_{res} are the initial and the residual masses of the impactor, respectively.

Lambert and Jonas (1976) (see also Zukas (1982)) noted that most of the known models can be represented in the unified form given by Eq. (1) and found that this relation can be used for description of experimental data where a , n , v_{bl} are determined from the experiments and some theoretical considerations. Formula (1) was used by Anderson et al. (1996), Børvik et al. (1999), Czarnecki (1998), Hetherington and Rajagopalan (1996), Lee and Sun (1993) and Nennstiel (1999). Nixdorff (1983, 1984, 1987) showed that under certain assumptions theory by Awerbuch and Bodner (Awerbuch, 1970; Awerbuch and Bodner, 1974) implies Eq. (1) even for multilayered targets.

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Mileiko and Sarkisyan (1981) demonstrated that a solution of the equation of motion of the impactor yields Eq. (1) with $a = 1$ when a power-law dependence between the impactor drag force and its velocity is valid. Ben-Dor et al. (1998a) showed that this property is valid for the generalized power-law model taking into account, in particular, the loss or accumulation of the mass and change of the shape of the impactor during penetration. Since employing Eq. (1) is currently one of the preferred way to handle experimental data (Anderson et al., 1996), it is useful to understand the cause for the efficiency of this correlation. The main goal of the present study is to address this problem by constructing the most wide class of physically realistic models describing penetration which imply Eq. (1).

At first, we consider a relatively simple class of impactor–armor localized interaction models which comprise some unknown functions determining the influence of the shape and the velocity of the impactor on the drag force during penetration. For conical impactors this approach allows us to determine analytically such functions that imply Eq. (1). Then a generalization of the above model taking into account the change of impactor's mass, shape, etc. is considered which also implies Eq. (1).

2. Localized interaction model: conical impactor

Consider penetration of an impactor into an armor with a thickness b . In this section we assume that the mass of the impactor m and its shape (in particular, the initial length L) do not change during penetration. All the other notations are presented in Fig. 1. The axis h is associated with the target and the depth of the penetration is characterized by the coordinate of the impactor's base. At the beginning of the penetration $h = -L$, and at the moment of perforation $h = b$. The axis x is associated with the impactor. The part of the impactor between the planes $x = x_-(h)$ and $x = x_+(h)$ is located inside the armor and interacts with it. The subsequent analysis does not use special expressions for the functions $x_-(h)$ and $x_+(h)$. We assume here also that the impactor–armor interaction at a given location at the surface of the impactor which is in a contact with the armor can be described by the following equation:

$$d\vec{F} = \Omega(u, v) \vec{n}^0 dS, \quad u = \cos \beta = -\vec{v}^0 \cdot \vec{n}^0 \quad (3)$$

where $d\vec{F}$ is the force acting at the surface element dS of the impactor along the inner normal vector \vec{n}^0 at a given location at the surface of the impactor, \vec{v}^0 is the unit local velocity vector, β is the angle between the vector \vec{n}^0 and the vector $(-\vec{v}^0)$ and the function Ω determines a model of impactor–armor interaction. More details concerning these models and the examples of their application in impact dynamics can be found in Ben-Dor et al. (1997, 1998a,b,c, 1999, 2000), Bunimovich and Dubinsky (1995), Jones and Rule (2000) and Recht (1990) and references therein.

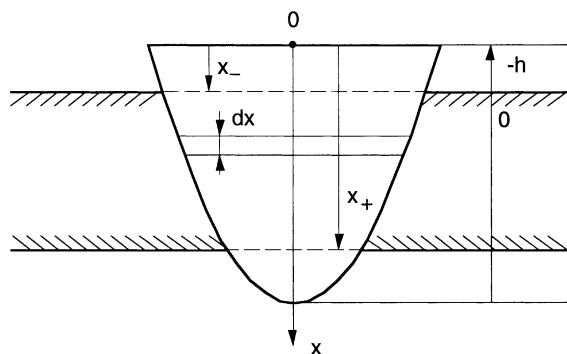


Fig. 1. Coordinates and notations.

The total force is determined by integrating the local force over the impactor–armor contact surface S between the planes $x = x_-(h)$ and $x = x_+(h)$. Using Eq. (3) the following expression can be obtained for the drag force of the conical impactor (for details see Ben-Dor et al. (1998b, 2000):

$$D = -\bar{v}^0 \int \int_S d\vec{F} = k_0 \psi(h) \omega(h) \quad (4)$$

where

$$k_0 = \pi \sin^2 \vartheta \tan^2 \vartheta, \quad \beta = \pi/2 - \vartheta, \quad \psi(h) = [x_+(h)]^2 - [x_-(h)]^2 \quad (5)$$

ϑ is the apex half angle of the conical impactor; since $u = \sin \vartheta = \text{const}$, function $\Omega(u, v)$ is replaced, for convenience, by the function $\omega(v)$.

Considering the velocity of the impactor v as a function of h the equation of its motion can be written as $mv(dv/dh) = -D$. Taking into account Eq. (4) and the initial conditions $v(-L) = v_{\text{imp}}$ its solution can be represented in the implicit form:

$$f(v_{\text{imp}}) - f(v) = \chi(h) \quad (6)$$

where

$$f(v) = \int_0^v \frac{z dz}{\omega(z)}, \quad \chi(h) = \frac{k_0}{m} \int_{-L}^h \psi(z) dz \quad (7)$$

Substituting $h = b$, $v = v_{\text{res}}$ into Eq. (6) we obtain the correlation between the impact and the residual velocities:

$$f(v_{\text{imp}}) - f(v_{\text{res}}) = \chi(b) \quad (8)$$

The ballistic limit velocity b_{bl} is defined as the impact velocity of the impactor required in order to emerge from the armor with a zero velocity. Substituting $v_{\text{imp}} = b_{\text{bl}}$, $v_{\text{res}} = 0$ into Eq. (6) and taking into account that $f(0) = 0$ we obtain:

$$f(v_{\text{bl}}) = \chi(b) \quad (9)$$

Eqs. (8) and (9) imply the correlation between the impact, the residual and the ballistic limit velocities:

$$f(v_{\text{imp}}) - f(v_{\text{res}}) = f(v_{\text{bl}}) \quad (10)$$

The latter expression depends on the function f that determines the model of impactor–armor interaction. Our goal is to find such class of functions f that Eq. (10) has the form of Eq. (1) with some parameters a and n .

Let us substitute v_{imp} from Eq. (1) into Eq. (10).

$$f\left[(av_{\text{res}}^n + v_{\text{bl}}^n)^{1/n}\right] - f(v_{\text{res}}) = f(v_{\text{bl}}) \quad (11)$$

Eq. (11) is a functional equation for determining the function f , and it must be valid for all $v_{\text{res}} \geq 0$. Let us transform Eq. (11) into a more convenient form:

$$\Phi(ay + v_{\text{bl}}^n) - \Phi(y) = \Phi(v_{\text{bl}}^n) \quad (12)$$

where

$$\Phi(z) = f(z^{1/n}), \quad y = v_{\text{res}}^n \quad (13)$$

We look for the solution of Eq. (12) in the following form:

$$\Phi(z) = \mu_1 z + \mu_2 \ln(\mu_3 z + 1) \quad (14)$$

where μ_1, μ_2, μ_3 are the unknown coefficients and $\Phi(0) = 0$. It is easy to show by substituting the expression for the function Φ from Eq. (14) into Eq. (12), that the following conditions must be satisfied:

$$\mu_1(a-1) = 0, \quad \mu_2\mu_3(a-1-\mu_3v_{bl}^n) = 0 \quad (15)$$

Eq. (15) has two solutions:

$$\mu_1 = 0, \quad a \neq 1, \quad \mu_3 = \frac{a-1}{v_{bl}^n}, \quad \mu_2 - \text{arbitrary} \quad (16)$$

and

$$\mu_1 \neq 0, \quad a = 1, \quad \mu_2\mu_3 = 0 \quad (17)$$

Let us consider the solution given by Eq. (16). In the latter case

$$\Phi(z) = \mu_2 \ln(\mu_3 z + 1), \quad f(z) = \mu_2 \ln(\mu_3 z^n + 1) \quad (18)$$

and Eq. (7) allows us to determine function $\omega(v)$:

$$\omega(v) = \frac{v}{df(v)/dv} = \sigma_1 v^2 + \sigma_2 v^{2-n} \quad (19)$$

where

$$\sigma_1 = \frac{1}{n\mu_2}, \quad \sigma_2 = \frac{1}{n\mu_2\mu_3} \quad (20)$$

Using Eqs. (9) and (16) and (20) the parameters a and v_{bl} can be expressed in terms of σ_1, σ_2 and n :

$$a = \exp(kn\sigma_1), \quad v_{bl} = \left[\frac{(a-1)\sigma_2}{\sigma_1} \right]^{1/n}, \quad k = \chi(b) \quad (21)$$

It can be shown that Eq. (19) includes also the solution given by Eq. (17) with $\sigma_1 = 0$ and

$$a = 1, \quad v_{bl} = (kn\sigma_2)^{1/n} \quad (22)$$

Eq. (22) can be represented in the form of Eq. (21) when $\sigma_1 \rightarrow 0$.

Thus, the model determined by Eqs. (3) and (19) with some parameters $\sigma_1 \geq 0, \sigma_2 > 0, n \geq 0$ (generally, they depend on β) implies Eq. (1) with parameter a determined by Eq. (21) which does not depend on σ_2 . Notably, Eq. (19) includes most of the known approximate models, namely, the models surveyed by Recht (1990) ($\sigma_1 > 0, \sigma_2 > 0, n = 2$), the model by Vitman and Stepanov (1959) ($\sigma_1 > 0, \sigma_2 > 0, n \neq 2$), the model by Mileiko and Sarkisyan (1981) ($\sigma_1 = 0, \sigma_2 > 0, n \neq 2$), etc.

The case $\sigma_1 = 0, \sigma_2 > 0, n = 2$ is associated with the assumption that the energy absorbed by the armor does not depend on the impact velocity, the case $\sigma_1 = 0, \sigma_2 > 0, n = 1$ implies a similar assumption concerning the momentum.

3. Generalized class of models

On the basis of the previous analysis we will show now that there exist more complex models which imply Eq. (1). These models are based on the following assumptions:

(1) During penetration the impactor can change its shape and accumulate and/or lose mass. The rate of change of mass is a function of the depth of the penetration and does not depend upon the impactor's velocity, i.e.,

$$m = m(h), \quad m^- = m^-(h), \quad m^+ = m^+(h) \quad (23)$$

where m is the mass of the impactor, m^+ and $(-m^-)$ are mass accumulation and mass loss from the beginning of motion, respectively. It is assumed that

$$v^+ = 0, \quad v^- = v \quad (24)$$

where v^+ is the velocity of the accumulated particles and v^- is the velocity of the lost particles.

(2) The differential drag force dD acting on the impactor's surface element between sections x and $x + dx$ (see Fig. 1) can be represented by the following formula:

$$dD = G_1(x, h)v^2 + G_2(x, h)v^{2-n} \quad (25)$$

with G_1, G_2 are non-negative functions and parameter $n \geq 0$.

(3) The part of the impactor interacting with the target depends only on the depth of the penetration, i.e.,

$$x_- = x_-(h), \quad x_+ = x_+(h) \quad (26)$$

The following equations are valid at the initial and final moments of the penetration, respectively:

$$x_-(-L) = x_+(-L) = L, \quad x_-(b) = x_+(b) = 0 \quad (27)$$

(4) The impactor can change its shape during penetration. Generally, the functions $m, m^-, m^+, G_1, G_2, x_-, x_+$ depend on the instantaneous impactor's shape, i.e., they account for deformation, accumulation and loss of mass of the impactor.

The equation of motion of a body with a variable mass can be written as (see, e.g., Corben and Stehle (1994)):

$$m\dot{v} + (v - v^+)\dot{m}^+ + (v - v^-)\dot{m}^- = -D \quad (28)$$

where the expression for the drag force D can be obtained from Eqs. (25) and (26):

$$D = g_1(h)v^2 + g_2(h)v^{2-n} \quad (29)$$

and

$$g_v(h) = \int_{x_-(h)}^{x_+(h)} G_v(x, h) dx, \quad v = 1, 2 \quad (30)$$

The model also comprises the equation of mass balance:

$$m(h) = m_{\text{imp}} + m^+(h) + m^-(h) \quad (31)$$

After substituting D from Eq. (29), v^+ and v^- from Eq. (24) and using change of variables $d/dt = v d/dh$, Eq. (28) can be written as follows:

$$\frac{dv}{dh} + \frac{1}{n} \varphi_1(h)v + \frac{1}{n} \varphi_2(h)v^{1-n} = 0 \quad (32)$$

where

$$\varphi_1(h) = \frac{n}{m(h)} \left[\frac{dm^+}{dh} + g_1(h) \right], \quad \varphi_2(h) = \frac{ng_2(h)}{m(h)} \quad (33)$$

Eq. (32) is a Bernoulli's equation that can be transformed (Kamke, 1959) to the linear equation with respect to the function $w = v^n$. Then after some algebra we can determine the solution of Eq. (32) with the initial condition $v(-L) = v_{\text{imp}}$:

$$[v(h)]^n = P_1(h) \left[v_{\text{imp}}^n - P_0(h) \right] \quad (34)$$

where

$$P_1(h) = \exp \left[- \int_{-L}^h \varphi_1(\xi) d\xi \right], \quad P_0(h) = \int_{-L}^h \frac{\varphi_2(\xi)}{P_1(\xi)} d\xi \quad (35)$$

For $h = b$ Eq. (34) reduces to

$$v_{\text{res}}^n = P_1(b) \left[v_{\text{imp}}^n - P_0(b) \right] \quad (36)$$

Substituting $v_{\text{imp}} = v_{\text{bl}}$ and $v_{\text{res}} = 0$ into Eq. (36) we obtain:

$$v_{\text{bl}} = [P_0(b)]^{1/n} \quad (37)$$

Eqs. (36) and (37) imply the correlation (1) with v_{bl} determined by Eq. (37) and

$$a = \frac{1}{P_1(b)} = \exp \left\{ n \int_{-L}^b \frac{1}{m(h)} \left[\frac{dm^+}{dh} + g_1(h) \right] dh \right\}$$

or using Eq. (31)

$$a = \left(\frac{m_{\text{res}}}{m_{\text{imp}}} \right) \exp \left\{ n \int_{-L}^b \frac{1}{m(h)} \left[\frac{dm^+}{dh} + g_1(h) \right] dh \right\} \quad (38)$$

Note that the parameter a does not depend upon the function $g_2(h)$.

The particular case when $g_1 = 0$ and the impactor can accumulate mass ($m^+ > 0$, $dm^+/dh > 0$, $m^- = 0$) or lose mass ($m^+ = 0$, $m^- < 0$, $dm^-/dh < 0$) was studied by Ben-Dor et al. (1998a). In the first case the value a is determined by Eq. (2), and in the second case $a = 1$. In both the cases a parameter a is independent on the history of mass change determined by the functions m^+ and m^- . Eq. (38) also shows that in the general case when $g_1 > 0$, $a > (m_{\text{res}}/m_{\text{imp}})^n$.

4. Concluding remark

We found a class of models describing penetration phenomenon which implies the well known Lambert–Jonas correlation between the impact, the residual and the ballistic limit velocities. The obtained results elucidate the reason for the validity of Lambert–Jonas correlation in various penetration problems.

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